ON A REMARKABLE PROPERTY BELONGING TO SOME CUBICS.

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THE well known theorem, Eucl. III, 31, can be enunciated as follows: If through a point on the circumference of a circle two chords be drawn, making with each other a right angle, the straight line joining the extremities of the chords will pass through a fixed point, the centre of the circle.

Can a Cubic be found having the same, or analogous property? This question will be partially answered in the following discussion.

A straight line drawn through a point, or a cubic, will generally meet the curve in two other points. Let the investigation therefore be limited to that class of *symmetrical* cubics which are represented by the equation

$$y^2 = x^2\,\frac{Ax+B}{Cx+D}, \tag{1}$$

the point through which the chords are to be drawn being the origin.

The equation of a straight line passing through the origin, and making an angle θ with the axis of x is

$$y = x \operatorname{tang} \theta.$$

The coordinates of the point of intersection between (1) and (2) are

$$\left.\begin{aligned} x_1 &= \frac{D\tan^2\theta - B}{A - C\tan^2\theta}, \\ y_1 &= x_1 \tan\theta. \end{aligned}\right\} \tag{3}$$

Let α be the angle included between the two chords, the coordinates of the point of intersection between the other end and the cubic are

$$\left.\begin{aligned} x_2 &= \frac{D\tan^2(\alpha-\theta) - B}{A - C\tan^2(\alpha-\theta)}, \\ y_2 &= -x_2 \tan(\alpha-\theta). \end{aligned}\right\} \tag{4}$$

The equation of the straight line passing through (3) and (4) is

$$y - y_1 = \frac{y_1 - y_2}{x_1 - x_2}(x - x_1). \tag{5}$$

If this line shall pass through a fixed point, that point must be on the axis of x because of the symmetrical form of the cubic. Then, putting $y = 0$ in the equation (5), we find the intersection between that line and the axis of x to be given by the abscissa

$$x = \frac{x_1y_2 - x_2y_1}{y_2 - y_1} = \frac{x_1x_2[\tan\theta + \tan(\alpha-\theta)]}{x_1\tan\theta + x_2\tan(\alpha-\theta)}, \ (6), =$$

$$\frac{[D\tan^2\theta - B][D\tan^2(\alpha-\theta) - B]}{AD[\operatorname{tg}^2\theta - \operatorname{tg}\theta\operatorname{tg}(\alpha-\theta) + \operatorname{tg}^2(\alpha-\theta)] - CD\operatorname{tg}^2\theta\operatorname{tg}^2(\alpha-\theta) + BC\operatorname{tg}\theta\operatorname{tg}(\alpha-\theta) - AB}$$

In order that this value may be independent of θ, $\frac{dx}{d\theta}$ must be equal to zero. Substituting

$$\tan\theta\tan(a—\theta) = t, \tag{7}$$

we get

$$\frac{dx}{d\theta}=\frac{[t^2(3D^2+2D^2\text{tg}^2a—BD\,\text{tg}^2a)+(2Dt—B)(B—D\,\text{tg}^2a)](BC—AD)}{[t^2(AD\,\text{tg}^2a—CD)+t(BC—2AD\,\text{tg}^2a—3AD)+(AD\,\text{tg}^2a—AB)]^2}\cdot\frac{dt}{d\theta}, \tag{8}$$

where the value of $dt\div d\theta$ is given by the equation

$$\frac{dt}{d\theta}=\frac{\sin(2a—2\theta)—\sin 2\theta}{2\cos^2\theta\cos^2(a—\theta)}. \tag{9}$$

Rejecting the solutions which will transform (1) into the equation of a conic, we see that $dx\div d\theta$ can be made equal to zero, first, if

$$t^2(3D^2+2D^2\tan^2a—BD\tan^2a)+(2Dt—B)(B—D\tan^2a) = 0, \tag{10}$$

which implies that the following eq's must be simultaneously satisfied;

$$\left.\begin{aligned}3D^2+2D^2\tan^2a—BD\tan^2a &= 0\\ D(B—D\tan^2a) &= 0\\ B(B—D\tan^2a) &= 0\end{aligned}\right\} \tag{11}$$

The only acceptable solution of these equations is

$$\left.\begin{aligned}B &= 3D,\\ a &= 60^\circ\ (120^\circ).\end{aligned}\right\} \tag{12}$$

Then the equation of the cubic will be

$$y^2 = x^2\frac{Ax+3D}{Cx+D}, \tag{A'}$$

and having drawn two chords through the origin, making with each other an angle of 60° (120°), the straight line joining the extremities of the ch'ds will pass through a fixed point on the axis of x situated at a distance from the origin, which from equation (6) we find to be

$$x = \frac{8D}{C—3A}. \tag{B'}$$

Secondly, $dx\div d\theta$ can be made equal to zero, if

$$\frac{dt}{d\theta} = 0. \tag{13}$$

Hence it follows that

$$a = 90^\circ. \tag{14}$$

From (6) we then deduce

$$x = \frac{B^2—BD(\tan^2\theta+\cot^2\theta)+D^2}{AD(\tan^2\theta+\cot^2\theta—1)—CD—AD+BC}, \tag{15}$$

and

$$\frac{dx}{d\theta} = \frac{D(AD—BC)(D—B)(\sec^2\theta—\operatorname{cosec}^2\theta)}{[AD(\tan^2\theta+\cot^2\theta—1)—CD—AB+BC]^2}. \tag{16}$$

Rejecting as in the former case solutions which will transform (1) into a conic, we see that $dx \div d\theta$ is equal to zero if

$$D = B. \tag{17}$$

The eq'n of the cubic will then be

$$y^2 = x^2 \frac{Ax + B}{Cx + D}, \tag{A''}$$

and the straight line joining the extremities of two chords, drawn through the origin at right angles to each other, will pass through a fixed point on the axis of x, the abscissa of which, from eq'n (6) or (15) we find to be

$$x = -\frac{B}{A}. \tag{B''}$$

The fixed point can of course lie at an infinite distance from the origin; the right line joining the extremities of the chords will then be parallel to the axis of x.

Concerning the conics, where the constant angle is 90° and the point from which the chords are to be drawn can be taken anywhere on the curve, the investigation is of such a special interest that it deserves a separate treatise, which we hope to give in another number of this periodical.

INTEGRATION OF THE GENERAL EQUATION OF MOTION.

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THE first difficulty which the student encounters in reading Gauss's *Theoria Motus Corporum Celestium*, is found in Article 3, where

$$\alpha x + \beta y + r = \gamma$$

is given as the general equation of the conic sections.

The equation is, I believe, due to La Place, who gave a demonstration of it in the Mecanique Ce'leste, Book II, Chap. III; and therefore, for want of a better name, I shall take the liberty of calling it La Place's Equation to the Conic Sections. I purpose in this brief paper to give a short and easy demonstration of this equation and to discuss it with the view of ascertaining the significance of the constants α, β, γ.

Let us assume the general equations of motion which are

$$\frac{d^2x}{dt^2} = -F\frac{x}{r} \ldots (1), \text{ and } \frac{d^2y}{dt^2} = -F\frac{y}{r} \ldots (2),$$

where F is the force. If F vary as $1 \div r^2$, as is the case in nature, then $F = \mu \div r^2$, where μ is the unit of force at the unit of distance or the absolute force, and (1) and (2) become